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On Packing of Squares and Cubes

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ABSTRACT

The main result of the paper is the following: Suppose $x_1 \geqslant x_2 \geqslant \cdots$ are the sides of cubes in the k-dimensional space and $a_1, a_2, \ldots, a_k$ are the edges of a rectangular parallelepiped. It is possible to pack the cubes into the parallelepiped if $a_j \geqslant x_1$, $j = 1, 2, \ldots, k$ and

$$x_1^k + \prod_{j=1}^{k} (a_j - x_1) \geqslant V,$$

where V denotes the volume of the cubes.

1. In a recent paper by J.W. Moon and L. Moser [2] it was proved among other results that given any set of squares with sides $x_1 \geqslant x_2 \geqslant x_3 \geqslant \cdots$ and with total area 1 it is possible to pack them into a square of side

$$a = x_1 + \sqrt{1 - x_1^2}.$$

The aim of this paper is to extend the above result to square packing into rectangles and to higher dimensional cube packing into rectangular parallelepipeds. The proofs show that the "most natural" packing (i.e., packing in order of magnitude, layer after layer) yields already the estimate of Moon and Moser in the 2-dimensional case and the analogous results in higher dimensions.

One of the consequences of our Theorem 1 is that any set of k-dimensional cubes of total volume V can be packed into a k-dimensional cube of volume $2^{k-1}V$. By a modification of the proof we can also show that any set of k-dimensional cubes of total volume V can be located in any rectangular parallelepiped of volume $2^{k-1}V$ if each edge of the parallelepiped is greater than the largest cube edge. We include (Theorem 2) the proof of this statement for $k = 2$ only (Section 2). In Section 3 we state without proof the analogous result for parallelepiped covering by cubes.

In Section 4 we obtain a best possible result for rectangle packing into rectangles. In Section 5 as applications of our Theorem 1 we show (i) that all squares of sides 1/2, 1/3, 1/4,... can be packed into a square of side 5/6, which is best possible, and (ii) that all rectangles of size

$$\frac{1}{k} \times \frac{1}{k+1}, k = 1, 2, \ldots$$

can be packed into a square of side 31/30. It would be desirable to find the smallest value of $\epsilon \geqslant 0$ such that all these rectangles can be packed into a square of side $1 + \epsilon$. Whether $\epsilon > 0$ or $\epsilon = 0$ is an open question.

A somewhat related open problem was proposed recently in [1]: Is it possible to pack the squares of sides 1, 2, 3,..., 24 into a square of side $70(1^2 = 1^2$ and $1^2 + 2^2 + \cdots + 24^2 = 70^2$ are the only solutions of $1^2 + 2^2 + \cdots + m^2 = n^2)$?

Finally, it would be desirable to extend our results of Section 2 to parallelepiped packing into parallelepipeds in higher dimensions.

2. THEOREM 1. *Any set of k-dimensional cubes of sides $x_1 \geqslant x_2 \geqslant \cdots$ with total volume V can be packed into any k-dimensional rectangular parallelepiped of size $a_1 \times a_2 \times \cdots \times a_k$ if $a_j > x_1$, $j = 1, 2, \ldots, k$ and*

$$x_1{}^k + (a_1 - x_1)(a_2 - x_1) \cdots (a_k - x_1) \geqslant V. \tag{2.1}$$

This result is in certain cases best possible.

REMARKS. (i) The minimum volume V^* of the large parallelepiped can be calculated as follows: Denote $x_1 a_j^{-1} = \lambda_j$, then

$$V^* = a_1 a_2 \cdots a_k = \lambda_1^{-1} \cdots \lambda_k^{-1} x_1{}^k$$

and the packing is possible according to (2.1) if

$$x_1{}^k[(\lambda_1^{-1} - 1)(\lambda_2^{-1} - 1) \cdots (\lambda_k^{-1} - 1) + 1] = V,$$

or in other words if

$$V^* = \frac{V}{\lambda_1\lambda_2 \cdots \lambda_k + (1 - \lambda_1)(1 - \lambda_2) \cdots (1 - \lambda_k)}.$$

If $\lambda_1, \ldots, \lambda_k$ are small then V^* is just slightly larger than V, so the packing is efficient if the cubes are small. It is also easy to check that if $\lambda_1, \lambda_2, \ldots, \lambda_k$ are not larger than 1/2, we have $V^* \leqslant 2^{k-1}V$.

(ii) If $a_1 = a_2 = \cdots = a_k = a$ in Theorem 1, we have the interesting special case: Any set of k-dimensional cubes of sides $x_1 \geqslant x_2 \geqslant \cdots$ with total volume V can be packed into a k-dimensional cube of side

$$a = x_1 + \sqrt[k]{V - x_1^k}.$$

Since

$$x_1 + \sqrt[k]{V - x_1^k} \leqslant 2 \sqrt[k]{\frac{V}{2}} \qquad \text{for all} \quad 0 \leqslant x_1 \leqslant \sqrt[k]{V}$$

the cubes can always be packed into a cube of side $2\sqrt[k]{V/2}$ or in other words into a cube of volume $2^{k-1}V$. The case $k = 2$ gives Moon and Moser's result mentioned in the introduction.

PROOF: We include the proofs for $k = 2$ and $k = 3$ only. The induction steps for higher dimensions are completely analogous.

The case $k = 2$. As mentioned we pack in order of magnitude layer after layer (see Fig. 1). On the ν-th layer we can pack the squares of

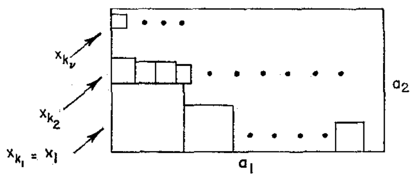

FIGURE 1

sides $x_{k_\nu}, x_{k_\nu+1}, \ldots, x_{k_{\nu+1}-1}$ where the integers k_ν are defined by $k_1 = 1$ and for $\nu = 1, 2, 3, \ldots$ by

$$a_1 - x_{k_{\nu+1}} < \sum_{j=k_\nu}^{k_{\nu+1}-1} x_j \leqslant a_1 . \tag{2.2}$$

Then for the area A packed into the ν-th layer we have by (2.2)

$$A_\nu = \sum_{j=k_\nu}^{k_{\nu+1}-1} x_j^2 \geqslant x_{k_\nu}^2 + (a_1 - x_{k_\nu})\, x_{k_{\nu+1}} - x_{k_{\nu+1}}^2 . \tag{2.3}$$

(If there are finitely many squares then $x_j = 0$ for $j > j_0$ but it does not affect our argument.) From (2.3) we obtain for the total area V

$$\begin{aligned} V = \sum_{\nu=1}^{\infty} A_\nu &\geqslant x_1^2 + \sum_{\nu=1}^{\infty} (a_1 - x_{k_\nu})\, x_{k_{\nu+1}} \\ &\geqslant x_1^2 + (a_1 - x_1) \sum_{\nu=1}^{\infty} x_{k_{\nu+1}} . \end{aligned} \tag{2.4}$$

Now by hypothesis, $x_1^2 + (a_1 - x_1)(a_2 - x_1) \geqslant V$, so from (2.4)

$$x_1 + \sum_{\nu=1}^{\infty} x_{k_{\nu+1}} \leqslant a_2 .$$

This proves the theorem in case $k = 2$.

The case $k = 3$. We again pack in order of magnitude layer after layer (see Fig. 2). Using the of result Theorem 1 for $k = 2$ it is easy to see

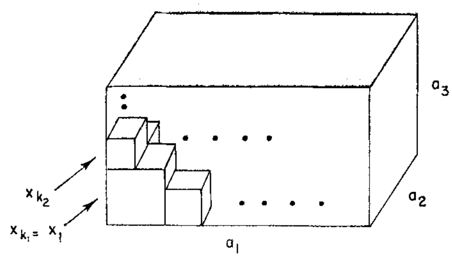

FIGURE 2

that on the ν-th layer we can pack the cubes of sides $x_{k_\nu}, \ldots, x_{k_{\nu+1}-1}$, where the integers k_ν are defined by $k_1 = 1$ and for $\nu = 2, 3, \ldots$ by

$$(a_1 - x_{k_\nu})(a_2 - x_{k_\nu}) - x_{k_{\nu+1}}^2 < \sum_{j=k_\nu}^{k_{\nu+1}-1} x_j^2 \leqslant (a_1 - x_{k_\nu})(a_2 - x_{k_\nu}) + x_{k_\nu}^2 . \tag{2.5}$$

Then for the volume V_ν of the ν-th layer we have by (2.5)

$$V_\nu = \sum_{j=k_\nu}^{k_{\nu+1}-1} x_j^3 \geqslant x_{k_\nu}^3 + (a_1 - x_{k_\nu})(a_2 - x_{k_\nu})\, x_{k_{\nu+1}} - x_{k_{\nu+1}}^3 ,$$

and for the total V

$$V = \sum_{\nu=1}^{\infty} V_\nu \geqslant x_1^3 + (a_1 - x_1)(a_2 - x_1) \sum_{\nu=1}^{\infty} x_{k_{\nu+1}} . \tag{2.6}$$

Combining (2.6) and the hypothesis (2.1) we have

$$x_1 + \sum_{\nu=1}^{\infty} x_{k_{\nu+1}} \leqslant a_3 ,$$

which proves the theorem in case $k = 3$. To see that the statement of Theorem 1 is in certain cases best possible, it is enough to consider the example

$$x_1 = x_2 = \cdots = x_{N+1} = \frac{1}{\sqrt[k]{N+1}},$$

where $N = n^k$, n an integer,

$$a_1 = a_2 = \cdots = a_{k-1} = \frac{n+1-\epsilon}{\sqrt[k]{n^k+1}}.$$

As $\epsilon \downarrow 0$ we see that the left-hand side of (2.1) approaches the right-hand side.

We show now that the area necessary for square packing into rectangles is never larger than twice the total area of the squares to be packed.

THEOREM 2. *Any set of squares of sides $x_1 \geqslant x_2 \geqslant \cdots$ with total area A can be packed (in order of magnitude, layer after layer) into any rectangle of size $a \times b$ if* $\min(a, b) \geqslant x_1$ *and $a \cdot b \geqslant 2A$.*

PROOF: Suppose $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_p \geqslant a/2 > x_{p+1} \geqslant \cdots$, then in the proof of Theorem 1 we have $x_{k_\nu} = x_\nu$ for $\nu = 1, 2,..., p$ and thus

$$A_\nu = x_\nu{}^2 \geqslant \frac{a}{2} x_\nu, \qquad \nu = 1, 2,..., p-1 \tag{2.7}$$

and by (2.3)

$$A_\nu \geqslant x_{k_\nu}^2 - x_{k_{\nu+1}}^2 + \frac{a}{2} x_{k_{\nu+1}}, \qquad \nu = p+1,... . \tag{2.8}$$

For $\nu = p$ we have

$$\begin{aligned} A_p &\geqslant x_p{}^2 - x_{k_{p+1}}^2 + (a - x_p)\, x_{k_{p+1}} \\ &= -x_{k_{p+1}}^2 + \frac{a}{2}(x_p + x_{k_{p+1}}) + \left(x_p - \frac{a}{2}\right)(x_p - x_{k_{p+1}}). \end{aligned} \tag{2.9}$$

From (2.7), (2.8), and (2.9):

$$A = \sum_{\nu=1}^{\infty} A_\nu \geqslant \frac{a}{2}\left(\sum_{\nu=1}^{p} x_\nu + \sum_{\nu=p}^{\infty} x_{k_{\nu+1}}\right).$$

By hypothesis $ab/2 \geqslant A$, we obtain

$$b \geqslant \sum_{\nu=1}^{\infty} x_{k_\nu},$$

which proves the statement.

3. For parallelepiped covering by cubes we have the following:

THEOREM 3. *Any set of k-dimensional cubes of sides $x_1 \geqslant x_2 \geqslant \cdots$ and with total volume V can cover any rectangular parallelepiped of size $a_1 \times a_2 \times \cdots \times a_k$ if*

$$(a_1 + x_1)(a_2 + x_1) \cdots (a_k + x_1) \leqslant V + x_1{}^k.$$

This estimate is in certain cases best possible.

COROLLARY (i). *Any set of k-dimensional cubes of sides $x_1 \geqslant x_2 \geqslant \cdots$ and with total volume V can cover a k-cube of side*

$$a = \sqrt[k]{x_1{}^k + V} - x_1 . \tag{3.1}$$

Since the cubes obviously can cover a cube of side x_1 we consider those values of x_1 for which $a \geqslant x_1$, i.e., $x_1{}^k + V \geqslant (2x_1)^k$. The minimum value of a then attained is

$$a = \sqrt[k]{\frac{V}{2^k - 1}}, \qquad \textit{if} \quad x_1 = \sqrt[k]{\frac{V}{2^k - 1}}.$$

In other words, the set of cubes can cover always a cube whose volume is $V/(2^k - 1)$.

COROLLARY (ii). *If $a_j \geqslant x_1$, $j = 1, 2, \ldots, k$, then*

$$-x_1{}^k + (a_1 + x_1) \cdots (a_k + x_1) \leqslant (2^k - 1) a_1 a_2 \cdots a_k$$

and thus by (3.1) *if the volume $a_1 \cdot a_2 \cdots a_k$ of the parallelepiped is not greater than $V/(2^k + 1)$ the covering is possible.*

4. For rectangle packing into rectangles we have the following result:

THEOREM 4. *Any set of rectangles with sides $\leqslant x_1$ and total area A can be packed into any rectangle of size $a \times b$ if $a \geqslant x_1$ and $a \cdot b \geqslant 2A + (a^2/8)$. This result is best possible.*

PROOF: We denote the larger sides of the rectangles to be packed by

$x_1 \geqslant x_2 \geqslant \cdots$, and the smaller sides by $y_1, y_2, \ldots$. Suppose $a \geqslant x_1$ and

$$x_1 \geqslant x_2 \geqslant \cdots \geqslant x_p \geqslant \frac{a}{2} > x_{p+1} \geqslant \cdots. \tag{4.1}$$

All rectangles of larger sides $x_{p+1}, x_{p+2}, \ldots$ we arrange in decreasing order of *smaller side*, which we denote by $z_1 \geqslant z_2 \geqslant z_3 \geqslant \cdots$ and their larger sides by $w_1, w_2, \cdots$ correspondingly. Now we pack the rectangles with larger sides $x_2, \cdots, x_p$ layer after layer and then the rectangles with smaller sides $z_1, z_2, \ldots$, layer after layer (see Fig. 3). If A_ν denotes the area packed into the ν-th layer we have

$$A_\nu = x_\nu y_\nu \geqslant \frac{a}{2} y_\nu, \qquad \nu = 1, 2, \ldots, p \tag{4.2}$$

$$A_{p+j} \geqslant w_{k_j} z_{k_j} + (a - w_{k_j}) z_{k_{j+1}} - w_{k_{j+1}} z_{k_{j+1}}, \qquad j = 1, 2, \ldots, \tag{4.3}$$

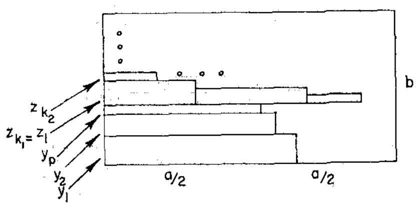

FIGURE 3

where $k_1 = 1$ and k_j is defined for $j \geqslant 2$ by

$$a - w_{k_{j+1}} < \sum_{\nu=k_j}^{k_{j+1}-1} w_\nu \leqslant a. \tag{4.4}$$

From (4.2) and (4.3) we have for the total area

$$A = \sum_{\nu=1}^{\infty} A_\nu \geqslant \frac{a}{2} \sum_{\nu=1}^{p} y_\nu + w_1 z_1 + \sum_{\nu=1}^{\infty} (a - w_{k_\nu}) z_{k_{\nu+1}}$$

and, since $w_{k_\nu} \leqslant a/2$ for all ν and $w_1 \geqslant z_1$,

$$A \geqslant \frac{a}{2} \sum_{\nu=1}^{p} y_\nu + \frac{a}{2} \sum_{\nu=1}^{\infty} z_{k_{\nu+1}} + z_1^2.$$

Thus

$$A - z_1^2 + \frac{a}{2} z_1 \geqslant \frac{a}{2} \left(\sum_{\nu=1}^{p} y_\nu + \sum_{\nu=1}^{\infty} z_{k_\nu} \right).$$

Now,

$$\frac{a}{2} z_1 - z_1^2 \leqslant \frac{a^2}{16} \qquad \text{for all} \quad 0 \leqslant z_1 \leqslant \frac{a}{2},$$

we have by (4.5)

$$\frac{1}{a}\left(2A + \frac{a^2}{8}\right) \geqslant \sum_{\nu=1}^{p} y_\nu + \sum_{\nu=1}^{\infty} z_{k_\nu}.$$

Since by hypothesis $2A + (a^2/8) \leqslant ab$ the theorem follows.

To show that the statement in full generality is best possible, we consider the special case of two rectangles with sides

$$x_1 = 1, y_1 = \epsilon, x_2 = y_2 = \frac{1}{4}$$

and for the rectangle to be packed in we choose $a = 1$. Then as it is easily seen we must have $b \geqslant 1/4 + \epsilon$. Our formula of Theorem 4 yields

$$ab = b \geqslant 2\left(\frac{1}{16} + \epsilon\right) + \frac{1}{8} = \frac{1}{4} + 2\epsilon.$$

5. Applications. (i) All squares of sides 1/2, 1/3, 1/4,... can be packed into a square of side 5/6. It is not difficult to see that this is the smallest square into which this can be done since the sum of side lengths of the two largest squares is $1/2 + 1/3 = 5/6$. It should be noted that the total area of the square to be packed is $(\pi^2/6) - 1 = 0.6449$ while $(5/6)^2 = 0.6944$, so that the surplus area is about 8 %.

We locate the first 19 squares denoted by 2, 3,..., 20 in Figure 4 as indicated. It is possible to pack the remaining squares in the unoccupied rectangle of size $17/60 \times 4/15$ since (see Theorem 1 for $k = 2$, $x_1 = 1/21$)

$$\left(\frac{1}{21}\right)^2 + \left(\frac{17}{60} - \frac{1}{21}\right)\left(\frac{4}{15} - \frac{1}{21}\right) > \frac{1}{20} > \sum_{\nu=21}^{\infty} \frac{1}{\nu^2}.$$

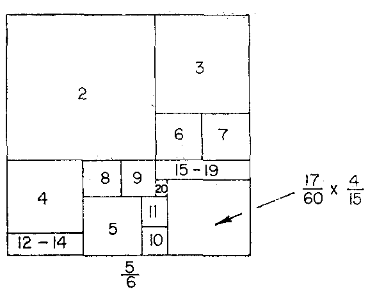

Figure 4

(ii) All rectangles of size $1/k \times 1/(k+1)$, $k = 1, 2, \ldots$ can be packed into a square of side $31/30$.

We locate the first 16 rectangles denoted by $1, 2, \ldots, 16$ as indicated in Figure 5. The remaining rectangles of size $1/17 \times 1/18$, $1/18 \times 1/19, \ldots$ can be packed into the unoccupied rectangle of size $1/2 \times 1/5$ since

$$\left(\frac{1}{17}\right)^2 + \left(\frac{1}{2} - \frac{1}{17}\right)\left(\frac{1}{5} - \frac{1}{17}\right) > \frac{1}{16} > \sum_{\nu=17}^{\infty} \frac{1}{\nu^2}$$

and thus according to Theorem 1 all squares of sides $1/17, 1/18, \ldots$ can be located there.

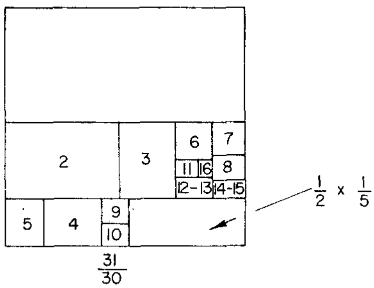

FIGURE 5

(iii) We are able to answer the question raised in [2]: What is the smallest number S such that any set of squares of total area 1 may be packed in a rectangle of base 1, height S?

By Theorem 2 we know that $S \leqslant 2$. On the other hand by considering four squares with sides

$$x_1 = x_2 = x_3 = \frac{1}{2} + \epsilon, \qquad x_4 = \sqrt{1 - 3\left(\frac{1}{2} + \epsilon\right)^2}$$

it is easy to see that $S > 2 - 0(\epsilon)$. Thus $S = 2$.

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2. J. W. MOON AND L. MOSER, Some Packing and Covering Theorems, *Colloq. Math.* **17** (1967), 103–110.